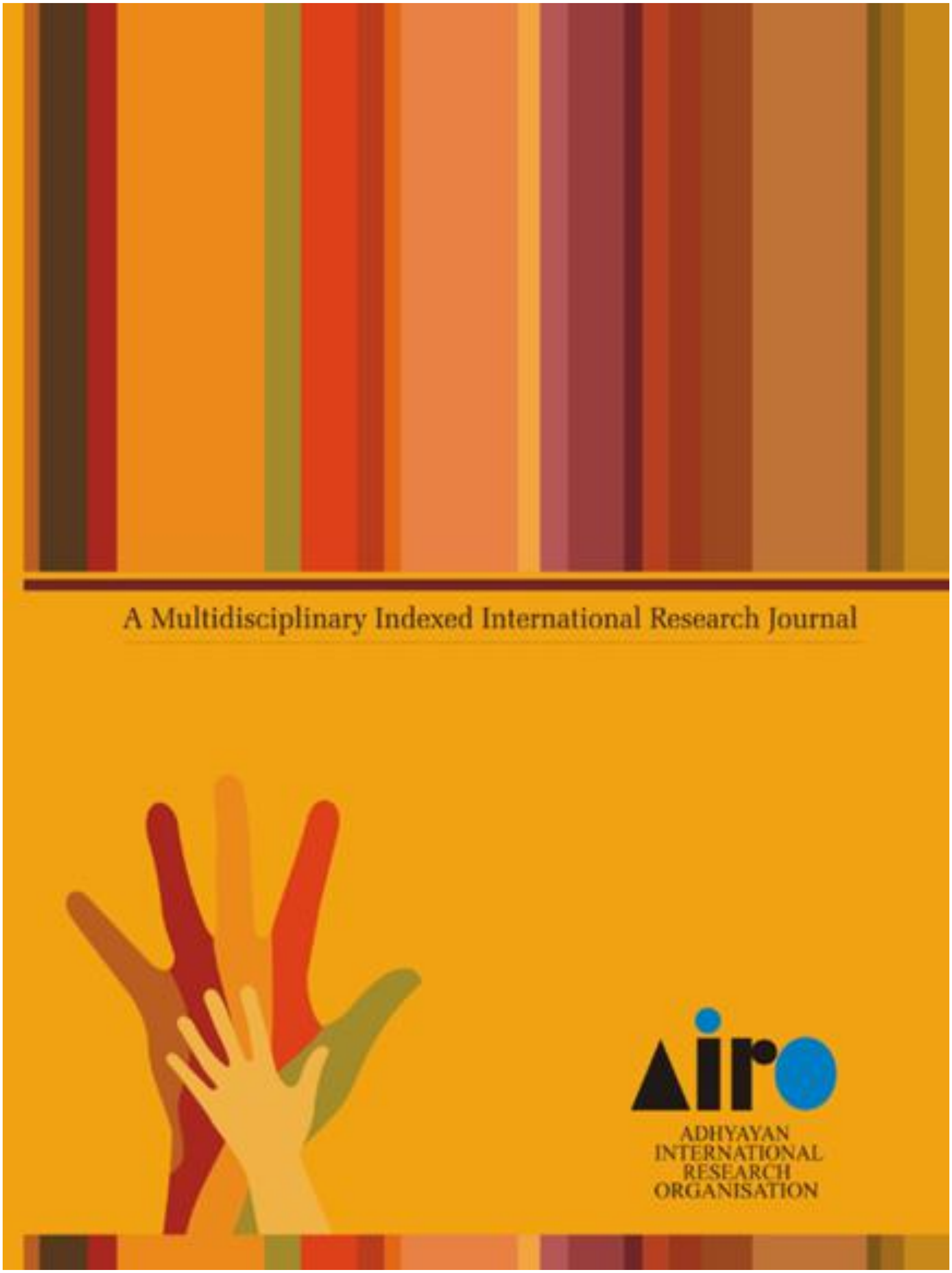
A Multidisciplinary Indexed International Research Journal
ADHYAYAN
INTERNATIONAL
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STUDY OF SOME ALGEBRAIC AND TOPOLOGICAL PROPERTIES IN DIFFERENCE SEQUENCE SPACE WITH FUZZY METRIC SPACE

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Abstract

In this paper we introduce the notion of difference operator $\Delta_m (m \geq 0$ an integer) for studying properties of some sequence space. We define the sequence spaces $c^F(\Delta_m)$, $c_0^F(\Delta_m)$, $\ell_\infty^F(\Delta_m)$ and study some topological properties like Solidness, Symmetricity and convergence free of this sequence space.

Key words: fuzzy Sequence, difference sequence, Fuzzy metric space, Solidness Symmetricity and Convergence free

Introduction

The Concept of fuzzy set was introduced by Zadeh [11]. Bounded and convergent sequence of fuzzy numbers were studied by Matloka [6], where it is shown that every convergent sequence is bounded. Later on different classes of sequences of fuzzy numbers have been studied by Esi [2], Tripathy and Nanda [10], Savas [7], Fang and Hang [3], Choudhury and Tripathy [1] and many others.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on real line R. For $X, Y \in D$, we define

$$d(X,Y) = \max\left(|a_1 - b_1|, |a_2 - b_2|\right) \text{ where } X = [a_1, a_2], Y = [b_1, b_2]$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is fuzzy set on R and is a mapping $X : R \to I(=[0,1])$ associating each real number t with its grade membership $X(t)$.

A sequence space E is said to be normal (or solid) if $(Y_k) \in E$, whenever $|Y_k| \le |X_k|$ for all $k \in N$ and $(X_k) \in E$.

For a sequence $x = (x_k)$, $S(x)$ denote the set of all permutation of the elements of (x_k) that is $S(x) = \{(x_{\pi(k)})\}$, where π denote permutation over N.

A sequence spaces E is said to be symmetric if $S(x) \subseteq E$, for all $x \in E$.

Let $K = \{k_1 < k_2 < k_3 \ . \ . \ . \ . \ \} \subseteq N$ and E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(X_{k_n}) \in w : (X_n) \in E\}$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E i.e.,Y is a canonical pre-image λ_K^E if and only if Y is a canonical pre-image of some $X \in \lambda_K^E$.

A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

From the above definition we have the following remark.

Remark 1. A sequence space E is solid $\Rightarrow$ E is monotone. It may refer to Kamthan and Gupta.

A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$, whenever $(x_k),(y_k) \in E$.

A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Fuzzy real numbers are considered as generalization of interval and multi-level interval numbers and so the knowledge of interval arithmetic is a basic requirement to introduce the fuzzy arithmetic. The following arithmetic operations will serve as basic knowledge for better understanding of the different operations of fuzzy number.

The α - level set of a fuzzy real number X, for $0 < \alpha \le 1$ denoted X^α is defined as $X^\alpha = \{t \in R : X(t) \ge \alpha\}$; for α, it is the closure of the strong 0-cut (i.e. closure of the set $\{t \in R : X(t) > 0\}$) . Throughout the article α means $\alpha \in (0,1]$ unless otherwise stated.

Consider the closed intervals $A = [a_1, a_2]$ and $B =[b_1, b_2]$. For the arithmetic operations on the closed intervals $A =[a_1, a_2]$ and $B = [b_1, b_2]$. The arithmetic operations on the closed intervals as follows

(i) $A+B= [a_1 +b_1, a_2+ b_2]$

(ii) $A – B = [a_1 - b_2, \; b_1 - a_2]$

(iii) $A \times B = [c_1, c_2]$

where $c_1 = \min \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$

and $c_2 = \max \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$

(iv) $1 \div A = \left[a_2^{-1}, a_1^{-1}\right]$, provided $0 \notin A$.

Let D denote the set of all closed and bounded intervals on R. The order relation between two intervals A and B as follows;

$$A \le B, \text{ if } a_1 \le b_1 \text{ and } a_2 \le b_2$$

A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) = \min \, (X(s), X(r))$ where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$ then the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon])$, for all $a \in I$ is open in the usual topology of R, where $I = [0,1]$

The set of all upper semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } \; t = r \\ 0, & \text{for } \; t \ne r \end{cases}$$

The absolute value , $|X|$ of $X \in R(I)$ is defined by (use for instance Kaleva and Seikkala [4])

$$|X|(t) = \begin{cases} \max(X(t), X(-t)), & \text{for } \; t \ge 0 \\ 0, & \text{for } \; t < 0 \end{cases}$$

A fuzzy real number is non negative if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real number is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X,Y) = \sup_{0\le\alpha\le 1} d(X^{\alpha}, Y^{\alpha})$$

It is well known that $(R(I), \bar{d})$ is a complete metric space. Then $\bar{d}$ defines a metric on $R(I)$.

DEFINITIONS AND PRELIMINARIES

After introduction of $R(I)$, different classes of fuzzy real valued sequences were introduced and studied by Tripathy [9] and others. Through the article w^F, c^F, c_0^F, ℓ_∞^F denote the classes of all, convergent, null and bounded sequences space of fuzzy real numbers.

A fuzzy real number X is non negative if $X(t) = 0$ if for all $t < 0$, we will denote the set of all non negative fuzzy number by G.

DEFINITIONS. Let X be any non empty set , $d: X \times X \to G$ and let the mapping $L, R: [0,1]\times[0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$

Then

$$\left[d(x,y)\right]_{\alpha} = \left[\lambda_{\alpha}(x,y), \rho_{\alpha}(x,y)\right]$$

For all $x, y \in X$ and $0 < \alpha \le 1$ and the quadruple (X, d, L, R) is called fuzzy metric space, if the following conditions are satisfied

(1) $d(x,y) = \bar{0}$ if and only if $x = y$

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$

(3) For all $x, y, z \in X$

(*i*) $d(x,y)(s+t) \ge L\left(d(x,z)(s), d(z,y)(t)\right)$

when $s \le \lambda_1(x,z)$, $t \le (z,y)$ and $s+t \le \lambda_1(x,y)$

(*ii*) $d(x,y)(s+t) \le R\left(d(x,z)(s), d(z,y)(t)\right)$

when $s \ge \lambda_1(x,z)$, $t \ge \lambda_1(z,y)$ and $s+t \ge \lambda_1(x,y)$

The last condition is the triangle inequality. It is known that the triangle inequality (3)(ii) with R = max is equivalent to the triangle inequality

$$\rho_\alpha(x,y) \le \rho_\alpha(x,z) + \rho_\alpha(z,y)$$

for $\alpha \in (0,1]$ and $x,y,z \in X$

Again the triangle inequality $3(i)$ with L = min is equivalent to the triangle inequality

$$\lambda_\alpha(x,y) \le \lambda_\alpha(x,z) + \lambda_\alpha(z,y)$$

for $\alpha \in (0,1]$ and $x,y,z \in X$.

Throughout the chapter we shall consider the fuzzy metric space $(X, d, \min, \max)$ unless otherwise stated.

It is known that in a fuzzy metric space $(X, d, \min, \max)$ the triangle inequality (*iii*) is equivalent to

$$d(x,y) \le d(x,z) + d(z,y)$$

Let (X, d, L, R) be a fuzzy metric space . A sequence $\{X_n\}$ is said to converge to X denoted by

$$\lim_{n\to\infty} X_n = X$$

If and only if

$$\lim_{n\to\infty} d(X_n, X) = \bar{0}$$

$$\text{i.e } \lim_{n\to\infty} \rho_\alpha(X_n, X) = 0, \quad \text{for } \alpha \in (0,1]$$

A sequence $\{X_n\}$ is said to be Cauchy sequence if

$$\lim_{m,n\to\infty} d(X_m, X_n) = \bar{0}.$$

$$\text{i.e } \lim_{m,n\to\infty} \rho_\alpha(X_n, X) = 0, \quad \text{for } \alpha \in (0,1]$$

In the fuzzy metric space (X, d, L, R) every convergent sequence is also a Cauchy sequence.

Kizmaz [5] defined the difference sequence space for the crisp set. This concept was further generalized by Tripathy and Esi [9] as follows.

Let $m \geq 0$, be an integer then $Z_1(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z_1\}$ for $Z_1 = \ell_\infty, c$ and c_0. where $\Delta_m x_k = x_k - x_{k+m}$ for all $k \in N$. For m = 1, the spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ are studied by Kizmaz [5].

The idea of Kizmaz [5] was applied by Savas [7] for introducing the notion of difference sequences for fuzzy real numbers and study their different properties.

We introduce the following difference sequence of fuzzy real numbers following Tripathy & Esi [9] as follows :

Let $m \geq 0$, be an integer then,

$$Z(\Delta_m) = \{(x_k) \in w^F : (\Delta_m x_k) \in Z_1\} \text{ where } Z_1 = \ell_\infty^F, c^F \text{ and } c_0^F$$

Main Result

Result 1 : Let d be a fuzzy metric space, then the classes of sequences $c^F(\Delta_m)$, $c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are not solid.

Proof of the result : We shall prove the result for $c^F(\Delta_m)$ and for other it can prove similarly. The result follows from the following example

Example 1:

Let m = 2,

Consider the sequence (X_n) defined by

$$X_n(t) = \begin{cases} \dfrac{nt+n+1}{n+1}, & \text{for } -1-\dfrac{1}{n} \leq t \leq 0 \\ \dfrac{n+1-nt}{n+1}, & \text{for } 0 \leq t \leq 1+\dfrac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\Delta_2 X_n(t) = \begin{cases} \dfrac{nt(t+2)+2n(n+2)+2n+2}{2n(n+2)+2n+2}, & \text{for } -2-\dfrac{1}{n}-\dfrac{1}{n+2} \le t \le 0 \\ \dfrac{2n(n+2)+2n+2-nt(t+2)}{2n(n+2)+2n+2}, & \text{for } 0 \le t \le 2+\dfrac{1}{n}+\dfrac{1}{n+2} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim\limits_{n\to\infty} \Delta_2 X_n = \Delta_2 X$

where $$\Delta_2 X_2(t) = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \le t \le 0 \\ \dfrac{2-t}{2}, & \text{for } 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{n\to\infty} \Delta_2 X_n = \Delta_2 X$$

where

$$\Delta_2 X(t) = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \le t \le 0 \\ \dfrac{2-t}{2}, & \text{for } 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Then for all $\alpha \in (0,1]$, we have

$$[\Delta_2 X_n]^\alpha = \left[\frac{(\alpha-1)\left[2n(n+2)+(2n+2)\right]}{n(n+2)}, \frac{\left[2n(n+2)+2n+2\right](1-\alpha)}{n(n+2)} \right]$$

and

$$[\Delta_2 X]^\alpha = [2(\alpha-1), 2(1-\alpha)]$$

Then clearly $(\Delta_2 X_n) \in c^F$.

Now consider the sequence of scalar (β_n) defined by

$$(\beta_n) = \begin{cases} 1, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

Then $(\beta_n X_n) = \left\{ X_1,\ \bar{0},\ X_3,\ \bar{0},\ X_5,\ \bar{0},\ .....\right\} \notin c^F(\Delta_2)$

Hence $c^F(\Delta_2)$ is not solid.

Result 2 : Let d be a fuzzy metric space, then the classes of sequences $c^F(\Delta_m)$, $c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are not symmetric.

Solution : Result follows from the following example

Example 2 : Let m = 1, and consider

$$A = \begin{cases} \dfrac{t+1}{1}, & \text{for } -1 \le t \le 0 \\ \dfrac{1-t}{1}, & \text{for } 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$B = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \le t \le 0 \\ \dfrac{2-t}{2}, & \text{for } 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$C = \begin{cases} \dfrac{t+3}{3}, & \text{for } -3 \le t \le 0 \\ \dfrac{3-t}{3}, & \text{for } 0 \le t \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$D = \begin{cases} \dfrac{t+4}{4}, & \text{for } -4 \le t \le 0 \\ \dfrac{4-t}{4}, & \text{for } 0 \le t \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$E = A - B = \begin{cases} \dfrac{t+3}{3}, & \text{for } -3 \le t \le 0 \\ \dfrac{3-t}{3}, & \text{for } 0 \le t \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

$$F = C - D = \begin{cases} \dfrac{t+7}{7}, & \text{for } -7 \le t \le 0 \\ \dfrac{7-t}{7}, & \text{for } 0 \le t \le 7 \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\alpha \in [0,1]$

$$E^{\alpha} = \left[3\alpha - 3, 3 - 3\alpha\right]$$

$$F^{\alpha} = \left[7\alpha - 7, 7 - 7\alpha\right]$$

Then consider the sequence

$$(X_n) = \left\{E,\ F,\ E,\ F,\ E,\ F\ \ldots\ldots\right\} \in c^{F}(\Delta)$$

Now consider the re- arrangement (Y_n) of the sequence (X_n) as following

$$(Y_n)=\{E,\ E,\ F,\ F,\ E,\ E\ .\ .\ .\ .\ .\ \}\notin c^F(\Delta)$$

Hence $c^F(\Delta)$ is not Symmetric.

Result 3 : Let d be a fuzzy metric space, then the classes of sequences $c^F(\Delta_m),\ c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are not Convergence free.

Solution : We shall prove the result for $c^F(\Delta_m)$ and for other it can prove similarly. The result follows from the following example

Example 3 : Let m = 2.

Consider the sequence (X_n) defined by

$$X_n(t)=\begin{cases}\dfrac{nt+n+1}{n+1}, & \text{for } -1-\dfrac{1}{n}\le t\le 0\\ \dfrac{n+1-nt}{n+1}, & \text{for } 0\le t\le 1+\dfrac{1}{n}\\ 0, & \text{otherwise.}\end{cases}$$

Now

$$\Delta_2X_n(t)=\begin{cases}\dfrac{nt(t+2)+2n(n+2)+2n+2}{2n(n+2)+2n+2}, & \text{for } -2-\dfrac{1}{n}-\dfrac{1}{n+2}\le t\le 0\\ \dfrac{2n(n+2)+2n+2-nt(t+2)}{2n(n+2)+2n+2}, & \text{for } 0\le t\le 2+\dfrac{1}{n}+\dfrac{1}{n+2}\\ 0, & \text{otherwise.}\end{cases}$$

Then $\lim\limits_{n\to\infty}\Delta_2X_n=\Delta_2X$

where $$\Delta_2X_2(t)=\begin{cases}\dfrac{t+2}{2}, & \text{for } -2\le t\le 0\\ \dfrac{2-t}{2}, & \text{for } 0\le t\le 2\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\lim_{n\to\infty} \Delta_2 X_n = \Delta_2 X$$

where

$$\Delta_2 X(t) = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \le t \le 0 \\ \dfrac{2-t}{2}, & \text{for } 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Then for all $\alpha \in (0,1]$, we have

$$\left[\Delta_2 X_n\right]^{\alpha} = \left[\frac{(\alpha-1)\left[2n(n+2)+(2n+2)\right]}{n(n+2)}, \frac{\left[2n(n+2)+2n+2\right](1-\alpha)}{n(n+2)}\right]$$

and

$$\left[\Delta_2 X\right]^{\alpha} = \left[2(\alpha-1), 2(1-\alpha)\right]$$

Then clearly $(\Delta_2 X_n) \in c^F$.

Again consider the sequence (Y_n) defined by

$$Y_n(t) = \begin{cases} \dfrac{t+n}{n}, & \text{for } -n \le t \le 0 \\ \dfrac{n-t}{n}, & \text{for } 0 \le t \le n \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_2 Y_n(t) = \begin{cases} \dfrac{t+2n+2}{2n+2}, & \text{for } -2n-2 \le t \le 0 \\ \dfrac{2n+2-t}{2n+2}, & \text{for } 0 \le t \le 2n+2 \\ 0, & \text{otherwise} \end{cases}$$

Then for all $\alpha \in (0,1]$, we have

$$[\Delta_2 X]^{\alpha} = [(2n+2)(\alpha-1),(2n+2)(1-\alpha)]$$

Therefore $(\Delta_2 X_n) \notin c^F$.

Hence $c^F(\Delta_2)$ is not convergence free

Conclusions : Following the notion of difference operator Δ_m, introduced by Tripathy and Esi [9], the difference sequences $c^F(\Delta_m)$, $c_0^F(\Delta_m)$, $\ell_\infty^F(\Delta_m)$ of fuzzy numbers have been introduced and study some topological properties like Solidness, Symmetricity and Convergence free with fuzzy metric spaces.

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